A Decomposition Algorithm for Two-Stage Stochastic Programs with Nonconvex Recourse Functions

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Joint work with Dr. Ying Cui (UC Berkeley)

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Two-stage stochastic programs

Two-stage stochastic linear programs

$$\min_{x \in X} \; c^\top x + \mathbb{E}_{\xi \sim \mathbb{P}}[V(x,\xi)],$$

where

$$V(x,\xi) \triangleq \min_{y} \quad q_\xi^\top y \qquad \text{s.t.} \quad W_\xi \, y \le h_\xi - T_\xi \, x$$

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Example: power systems planning

- x: (production from) traditional thermal plants
- ξ: renewable energy and demand
- y: backup fast-ramping generators
- q_ξ: unit cost of fast-ramping generators

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Deterministic equivalent: (finite scenarios)

$$\begin{array}{lllllll}
\min\limits_{x, y^1, \cdots, y^S} & c^\top x + \frac{1}{S} \sum\limits_{s=1}^{S} (q^s)^\top y^s & & & & \\
\text{s.t.} & T^1 x + W^1 y^1 & & & & \le h^1 \\
& T^2 x & + W^2 y^2 & & & \le h^2 \\
& \vdots & & \ddots & & \vdots \\
& T^S x & & & + W^S y^S & \le h^S
\end{array}$$

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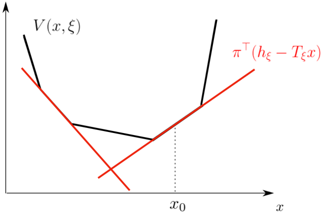
Key ideas of Benders decomposition:

1. $V(\bullet,\xi)$ is convex, piecewise affine.
2. $\partial V(\bullet,\xi)$ is easy to compute.



Benders optimality cuts

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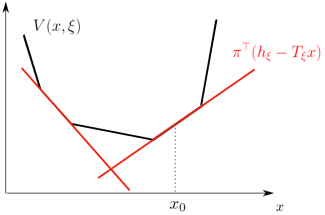

Benders optimality cuts

Question: What if V is NOT convex?

Two-stage stochastic programs

An example of **two-stage stochastic nonlinear programs**

$$\min_{x \in X} \; c^\top x + \mathbb{E}_{\xi \sim \mathbb{P}}[\, V(x, \xi)],$$

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$$V(x,\xi) \triangleq \begin{array}{rl} \min\limits_{y} & [\, q_\xi + D_\xi x \,]^\top y \\ \text{s.t.} & W_\xi \, y \le h_\xi - T_\xi \, x \end{array}$$

Motivation:

- Stochastic network interdiction[1]:

$$V(x,\xi) = \left[\begin{array}{rl} \max\limits_{y} & q_\xi^\top y \\ \text{s.t.} & W_\xi \, y \le h_\xi - T_\xi \, x \end{array} \right]$$

[1] Cormican, Morton and Wood, 1998

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Motivation:

- Power systems planning with decision-dependent costs:
  - x: traditional thermal plants
  - ξ: renewable energy and demand
  - y: backup fast-ramping generators

  Unit cost $q_\xi \to q_\xi + D_\xi x$ (reflect the ramp rate)

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Motivation:

- Decision-dependent probabilities[1]:

$$\mathbb{P}(\xi = \xi_1) = \frac{x'}{S}$$

[1] see, for example, [Hellemo, Barton and Tomasgard, 2018]

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$$\implies \quad \sum_{s=1}^{S} \mathbb{P}(\xi = \xi_s)\,(q^s)^\top y^s = \frac{1}{S} \sum_{s=1}^{S} \underbrace{S\,\mathbb{P}(\xi = \xi_s)\,(q^s)^\top}_{\text{linearly in } x'} y^s.$$

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This talk:

Can we develop a decomposition algorithm with convergence guarantee?

Nonconvex nonsmooth recourse functions

$$V(x) \triangleq \min_{y} \quad [\, q + D{\color{red}x} \,]^{\top} y \qquad \text{s.t.} \quad W\, y \le h - T{\color{red}x}$$

	linear $(D = 0)$	nonlinear
$\partial_C V(\bullet)$	easy to compute	
$V(\bullet)$	piecewise affine, convex	

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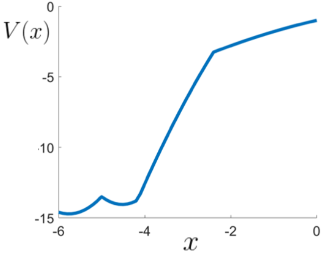

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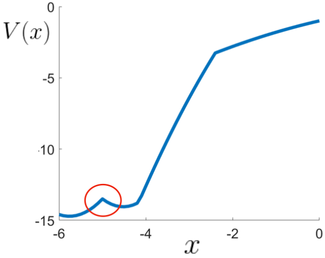

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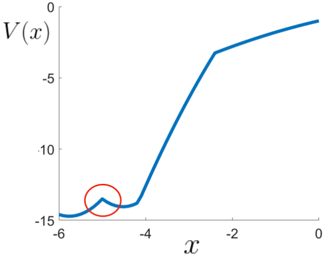

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"NOT Clarke-regular":

$$\sout{V(x) \ge \underbrace{V(\bar{x}) + a^\top(x - \bar{x})}_{\text{hyperplane generated at } \bar{x}} + o(\|x - \bar{x}\|), \quad \forall\, a \in \partial_C V(\bar{x}).}$$

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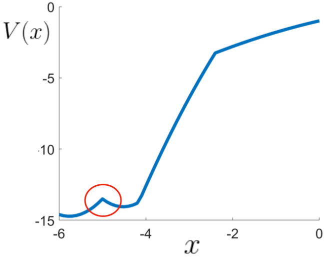

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Remedies: smoothing methods[1] or structured convex approximations

[1] Borges, Sagastizábal and Solodov, 2021

Perturbation analysis

Fix any $\bar{x}$.

1. Perturb the constraint:

$$V_{\mathsf{cvx}}(x) \triangleq \left[\begin{array}{ll} \min\limits_{y} & [\,q + D\bar{x}\,]^\top y \\ \mathrm{s.\,t.} & Wy \leq h - Tx \end{array}\right] \text{ is piecewise affine, convex.}$$

2. Perturb the objective:

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$\Longrightarrow$ Joint perturbations lead to nonconvex recourse functions:

$$V(x) \triangleq \left[\begin{array}{ll} \min\limits_{y} & [\,q + Dx\,]^\top y \\ \text{s.t.} & Wy \leq h - Tx \end{array} \right].$$

Lifting: implicit convexity-concavity

$$V(x) \triangleq \left[\begin{array}{ll} \min\limits_{y} & [\,q + Dx\,]^\top y \\ \text{s.t.} & Wy \leq h - Tx \end{array} \right]$$

$$\overline{V}(x, z) \triangleq \left[\begin{array}{ll} \min\limits_{y} & [\,q + Dz\,]^\top y \\ \text{s.t.} & Wy \leq h - Tx \end{array} \right]$$

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⇩

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implicitly convex-concave

$$\overline{V}({\color{red}x}, {\color{blue}z}) \triangleq \left[\begin{array}{ll} \min\limits_{y} & [\, q + D{\color{blue}z} \,]^\top y \\ \text{s. t.} & Wy \leq h - T{\color{red}x} \end{array} \right]$$

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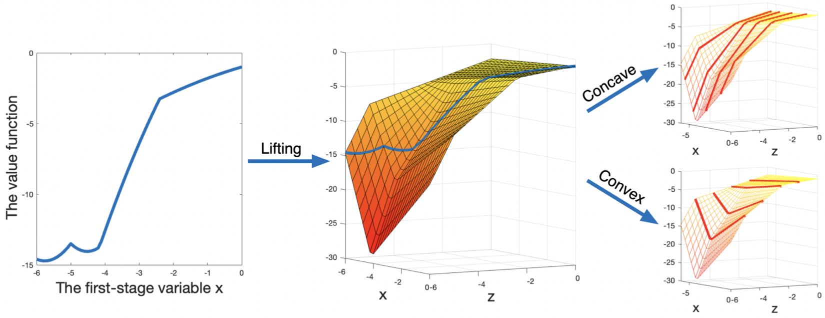

implicitly convex-concave in $\mathbb{R}$ convex-concave in $\mathbb{R}^2$ convex/concave in $\mathbb{R}$

Implicitly convex-concave: surrogations

The classical **Moreau envelope:**

$$e_\gamma V(x) \triangleq \inf_u \left\{ V(u) + \frac{\|u - x\|^2}{2\gamma} \right\}$$

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Benefit: $g_\gamma(x)$ and $\partial g_\gamma(x)$ can be evaluated by solving a convex problem.

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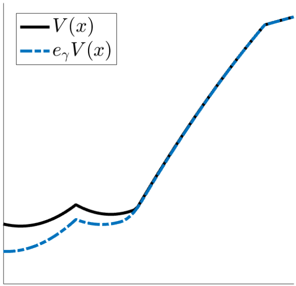

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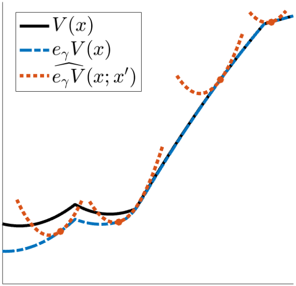
$$\implies e_\gamma V(x) \leq \boxed{\|x\|^2/(2\gamma) - \text{"linearization" of } g_\gamma \text{ at any point } x'} \triangleq \widehat{e_\gamma V}(x; x')$$



A decomposition algorithm: fixed scenarios

$$\min_{x \in X} \; c^\top x + \frac{1}{S} \sum_{s=1}^{S} V_s(x),$$

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Idea: $V_s \overset{(\gamma \downarrow 0)}{\longleftarrow} e_\gamma V_s \longleftarrow \widehat{e_\gamma V_s}$ (a strongly convex quadratic function)

Master problem:

$$\min_{x \in X} \; c^\top x + \frac{1}{S} \sum_{s=1}^{S} \widehat{e_\gamma V_s}(x; x^k)$$

$\widehat{e_\gamma V_s}$ (↑) x^{k+1} (↓)

Subproblems:

for each scenario, solve y to get $\widehat{e_\gamma V_s}(\bullet; x^k)$

(Need an outer loop to update $\gamma \downarrow 0$)

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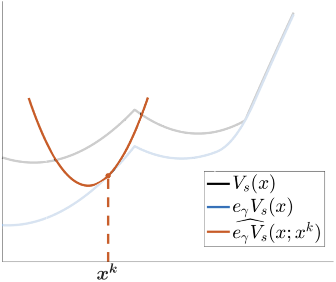

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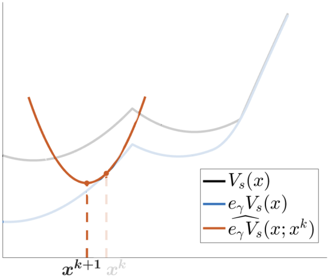


A decomposition algorithm: fixed scenarios

$$\min_{x \in X} \; c^\top x + \frac{1}{S} \sum_{s=1}^{S} V_s(x),$$

where

$$V_s(x) \triangleq \min_{y} \quad [\, q^s + D^s x \,]^\top y \qquad \text{s.t.} \quad W^s y \le h^s - T^s x$$

Idea: $V_s \overset{(\gamma \downarrow 0)}{\longleftarrow} e_\gamma V_s \longleftarrow \widehat{e_\gamma V_s}$ (a strongly convex quadratic function)

Master problem:

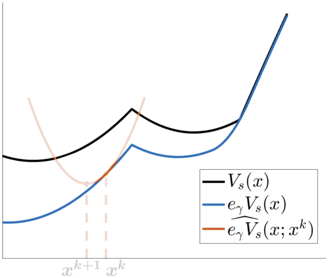
$$\min_{x \in X} \; c^\top x + \frac{1}{S} \sum_{s=1}^{S} \widehat{e_\gamma V_s}(x; x^k)$$

↑ $\widehat{e_\gamma V_s}$ ↓ x^{k+1}

Subproblems:

for each scenario, solve y to get $\widehat{e_\gamma V_s}(\bullet; x^k)$

(Need an outer loop to update $\gamma \downarrow 0$)



Main convergence results

Under technical conditions, we show that

Theorem (fixed scenarios)

(1) Any accumulation point is a (properly-defined) stationary point;

(2) If $\sum_{k=0}^{\infty} \gamma_k < +\infty$, then the sequence of objective values converges.

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Extension:
can be combined with the internal sampling under the sample-size requirement

$$\sum_{k=1}^{\infty} \frac{|S_{k+1}| - |S_k|}{|S_{k+1}|\,|S_k|^{\eta}} < +\infty \text{ for some } \eta \in (0, 1/2), \quad \text{e.g. } S_k = k.$$

Numerical experiments

A power system planning problem with decision-dependent probabilities[2]

- 1st stage $x \in \mathbb{R}^{10}$ and 2nd stage $y^s \in \mathbb{R}^{40}$.
- A nonconvex quadratic program in $(x, y^1, \cdots, y^S)$.

Dimensions of the deterministic equivalent

S	problem sizes	
	rows	columns
1,000	13,000	40,010
5,000	65,000	200,010
10,000	130,000	400,010
30,000	390,000	1,200,010
80,000	1,040,000	3,200,010
110,000	1,430,000	4,400,010

[2] Hellemo, Barton and Tomasgard, 2018

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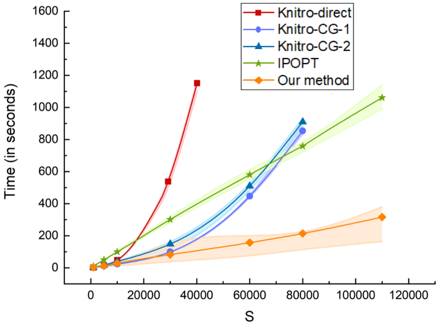

[2]Hellemo, Barton and Tomasgard, 2018

Extensions

Our algorithm can be applied to a broad class of recourse functions:

$$
\begin{aligned}
V(x) \triangleq \min_{y} \quad & f(x, y) \\
\text{s.t.} \quad & g(x, y) \leq 0
\end{aligned},
$$

which is implicitly convex-concave if $f(\bullet, \bullet)$ is concave-convex and $g(\bullet, \bullet)$ is convex.

Thank you!

This talk is based on the work:

- Hanyang Li, Ying Cui. *A decomposition algorithm for two-stage stochastic programs with nonconvex recourse functions*. SIAM Journal on Optimization, 2023