Variational Theory and Algorithms for a Class of Asymptotically Approachable Nonconvex Problems

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Joint work with Ying Cui (UC Berkeley)

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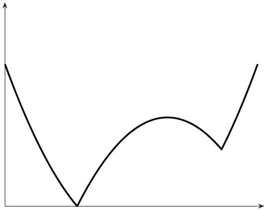
Amenable functions

Amenable: [Poliquin & Rockafellar, 1992] F has local representation

$$F = g \circ f = (\text{convex}) \circ (\text{smooth}).$$

Examples:

$$F(X) = \left\|\mathcal{A}\left(XX^\top\right) - b\right\|_1 \quad \leftarrow \quad \text{Phase retrieval, robust PCA, ...}$$



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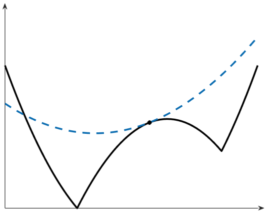
Facts:

- Convex approximation

$$F_{x^k}(x) \triangleq \underbrace{g\Big(f(x^k) + \nabla f(x^k)(x - x^k)\Big)}_{(\text{convex})\circ(\text{linear})} + \frac{\|x - x^k\|^2}{2\eta}$$



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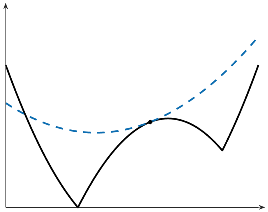
- Chain rule $\partial F(x) = \nabla f(x)^* \, \partial g\big(f(x)\big)$



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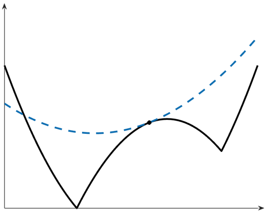

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Prox-linear algorithm for $\min_x F(x) \triangleq g(f(x))$:

$$\boxed{x^{k+1} = \operatorname{argmin}_x \; F_{x^k}(x)}$$

See [Fletcher, 1982, Burke & Ferris, 1995, Lewis & Wright, 2016, Drusvyatskiy & Paquette, 2019, ...]

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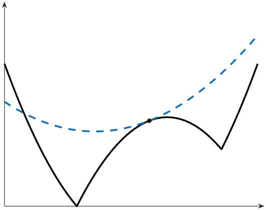

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Q: Can we go beyond amenable functions by relaxing the smoothness of F?

Beyond amenability

$$\min_{x \in X} F(x) \triangleq \varphi(f(x)),$$

where φ is convex and f is NOT continuously differentiable.

Motivating examples:

- Inverse optimal value problems [Ahmed & Guan, 2005]

$$f_i(x) \triangleq \begin{bmatrix} \min\limits_{y} & x^\top y \\ \text{s.t.} & A^i y \le b^i \end{bmatrix} \quad \forall\, i = 1, \cdots, m$$

  - x: unknown parameter of a linear program
  - $\{v_i\}_{1 \le i \le m}$: (noisy) observation of optimal values

  How to estimate the parameter x?

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$$\min_{x \in X} \sum_{i=1}^{m} |v_i - f_i(x)| = (\text{convex}) \circ (\text{nonsmooth})$$

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- Risk management with Value-at-risk:

$$\begin{aligned} \min_{x} \quad & \text{obj} \\ \text{s.t.} \quad & \mathrm{VaR}_{\alpha}(c(x, Z)) \geq r \end{aligned}$$

- $c(x, Z)$: the random profit of investments parameterized by decision x

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$$\delta_{[r,+\infty)}\Big(\mathrm{VaR}_\alpha(c(x, Z))\Big) = (\text{convex}) \circ (\text{nonsmooth})$$

$$\text{e.g. } Z = \begin{cases} 2 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/6 \\ -1 & \text{w.p. } 1/3 \end{cases} \quad \Rightarrow \quad \mathrm{VaR}_{1/3}(xZ + 1) = \min(2x + 1, -x + 1)$$

Asymptotic approachable problems

$$\min_{x \in X} \; F(x) \triangleq \varphi(f(x)),$$

where $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex, nondecreasing.

Structure assumption for f:

- Recall for smooth f with Lipschitz gradient:

$$f(x) - \left(f(x^k) + \nabla f(x^k)^\top (x - x^k)\right) = O(\|x - x^k\|^2)$$

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  - Suppose $f = g - h$ (g, h convex).

    $\left(\text{linearization of } g \text{ at } x^k\right) - h(x) \leq f(x) \leq g(x) - \left(\text{linearization of } h \text{ at } x^k\right)$

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  - Suppose f is approachable difference-of-convex (ADC):

    $$f = \lim_k \left(f^k \triangleq g^k - h^k\right) \qquad g^k, h^k \text{ convex.}$$

Asymptotic approachable problems

$$\min_{x \in X} F(x) \triangleq \varphi(f(x)) = (\text{convex nondecreasing}) \circ (\text{ADC}).$$

Ubiquity of ADC:

Theorem ([Royset, 2020])

$\forall$ lower semicontinuous function f, $\exists\{f^k\}$ such that $f = (\text{epi-})\lim_k f^k$, where

$$f^k(x) = \underbrace{\max_{1 \le i \le p_k} \left[\langle a^{k,i}, x\rangle + \alpha_i\right]}_{\text{convex}} - \underbrace{\max_{1 \le i \le q_k} \left[\langle b^{k,i}, x\rangle + \beta_i\right]}_{\text{convex}} \qquad \forall\, k.$$

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Ubiquity of ADC:

- Observation: Moreau envelope

$$e_{\lambda_k} f(x) \quad = \inf_u \left\{ f(u) + \frac{\|u-x\|^2}{2\lambda_k} \right\}$$

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- More examples: the optimal value functions, value-at-risk are nonsmooth but ADC!

Convex approximation and algorithms

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Need to solve

$$\min_{x \in X} F^k(x) \triangleq \varphi\Big(\underbrace{g^k(x) - h^k(x)}_{= f^k(x)} \Big) \qquad \forall\, k.$$

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Algorithm framework:

For $k = 1, 2, \cdots$

Solve $\min_x F^k(x)$ iteratively by its convex approximation

(within proper tolerance $\varepsilon^k \downarrow 0$)

Asymptotic stationary

$$\min_{x \in X} F(x) \triangleq \varphi(f(x)) = (\text{convex nondecreasing}) \circ (\text{ADC}) \qquad \text{(P)}$$

Convergence: Any accumulation point $\bar{x}$ generated by the algorithm satisfies

$$0 \in \partial\varphi\big(f(\bar{x})\big) \cdot \partial_A f(\bar{x}),$$ [1]

where $\partial_A f(x)$ is a "subdifferential" of f dependent on $\{f^k = g^k - h^k\}$:

$$\partial_A f(x) \triangleq \bigcup_{x^k \to x} \left\{\text{accumulation points of } \left[\partial g^k(x^k) - \partial h^k(x^k)\right]\right\}.$$

[1] This generalizes the stationary point in [Royset, 2022]

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Q: Is it a necessary condition for a local minimizer of (P)?

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Good or bad approximation? —Epi-convergence

Epi-convergence:

$$f^k \xrightarrow{e} f \quad \iff \quad \mathrm{Lim}_k \left(\mathrm{epi}\, f^k\right) = \mathrm{epi}\, f$$

[2] See monographs [Rockafellar & Wets, 2009], [Royset & Wets, 2022]

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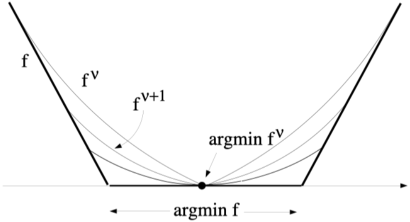
Epi-convergence:

$$f^k \xrightarrow{e} f \quad \Longleftrightarrow \quad \mathrm{Lim}_k \left(\mathrm{epi}\, f^k\right) = \mathrm{epi}\, f$$

Fact[2]**:** (global minimizers)

$$f^k \xrightarrow{e} f \quad \Longrightarrow \quad \left\{\text{accumulation points of } \left(\varepsilon^k\text{-argmin}\, f^k\right)\right\} \subset \mathrm{argmin}\, f, \quad \forall \varepsilon^k \downarrow 0.$$



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Necessary optimality condition under epi-convergence

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[3] NOT computationally friendly

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Idea:

Any local minimizer of (P) $\Longrightarrow$ A stationary point[3] for (P)

$\Updownarrow$

$$0 \in \partial\varphi\big(f(\bar{x})\big) \cdot \partial_A f(\bar{x})$$

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Extensions

Our analysis can be applied to

$$\min_x \sum (\text{univariate convex}) \circ (\text{ADC}).$$

Lemma (a monotonic decomposition)

For a univariate convex function φ, $\exists$ convex $\varphi^{\uparrow}$ and $\varphi^{\downarrow}$ such that

$$\varphi = \varphi^{\uparrow}(\text{nondecreasing}) + \varphi^{\downarrow}(\text{nonincreasing}).$$

Conclusion

- Beyond amenability: A class of asymptotically approachable problems

$$\min_x \sum (\text{univariate convex}) \circ (\text{ADC})$$

- Epi-convergence:

$$\Longrightarrow \quad \text{consistency of the global min} + \underbrace{\text{stationary points (subgradient-based)}}_{\text{necessary for the local min}}$$

Thank you!

This talk is based on the work:

- Hanyang Li, Ying Cui. *Variational Theory and Algorithms for a Class of Asymptotically Approachable Nonconvex Problems*. arXiv:2307.00780 (2023)

Appendix 1: Subgradient relationship under epi-convergence

Proposition

If f is ADC and $f^k (= g^k - h^k) \xrightarrow{e} f$, then

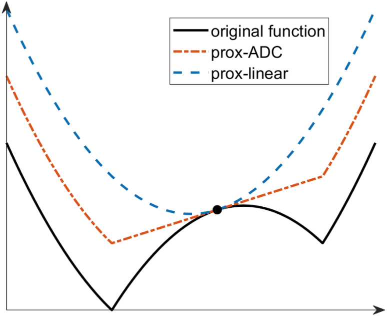
(a) $\partial f(x) \subset \partial_A f(x)$;

(b) $\partial f(x) \subset \partial_A f(x) \subset \operatorname{conv} \partial f(x)$ if f is locally Lipschitz continuous and $f^k = e_{\lambda_k} f$ for any $\lambda_k \downarrow 0$.

Appendix 2: Comparison with prox-linear method

$$\min_x \ \varphi(f(x))$$

φ is univariate convex, f is smooth with ℓ-Lipschitz gradient



$$f = \underbrace{\left(f + \frac{\ell}{2}\| \bullet \|^2\right)}_{\text{convex}} - \frac{\ell}{2}\| \bullet \|^2$$

Prox-ADC:

$$\varphi^{\uparrow}\left(f(x) + \frac{\ell}{2}\|x - x^k\|^2\right) + \varphi^{\downarrow}\left(f(x^k) + \nabla f(x^k)^\top (x - x^k) - \frac{\ell}{2}\|x - x^k\|^2\right)$$

Prox-linear:

$$\varphi\big(f(x^k) + \nabla f(x^k)^\top (x - x^k)\big)$$